

Chapter 11.3

11.3 Algebraic Extensions

Def A field extension $K \supset F$ is called algebraic when every element $u \in K$ is algebraic (over F).

Th 11.9 If $K \supset F$ is finite-dimensional then $K \supset F$ is algebraic.

Pf. Let $u \in K$. Needed: $f \in F[x]$ such that $f(u) = 0$.

Let $[K:F] = n$.

Consider $1_K, u, u^2, \dots, u^n \in K$ - $n+1$ elements of K .

Since $[K:F] = n$, the $n+1$ elements must be linearly dependent over F .

$c_0 \cdot 1_K + c_1 u + c_2 u^2 + \dots + c_n u^n = 0$, with $c_i \in F$

Let $f = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \in F[x]$. We have $f(u) = 0$.

Prop A simple extension $F(u) \supset F$ is algebraic iff u is algebraic over F .

Pf If u is transcendental, then $F(u) \supset F$ cannot be algebraic because $u \in F(u)$.

If u is algebraic (over F) then $F(u) \supset F$ is finite-dimensional, therefore algebraic.

Remark: $F(u) = \left\{ \frac{f(u)}{g(u)} \mid f, g \in F[x], g(u) \neq 0 \right\}$

Proved: together with u , every expression $\frac{f(u)}{g(u)}$ becomes algebraic

$$h(u^2 + u + 1) = 0, \quad h \in F[x]$$

$$\text{Ex } \mathbb{C} = \mathbb{R}(i)$$

$\frac{2+3i}{7+4i}$ - a root of some polynomial from $\mathbb{R}[x]$

Def Finitely generated extension

Start with an extension $K \supset F$, and $u_1, \dots, u_n \in K$

$F(u_1, \dots, u_n)$ - finitely generated extension

$$K \supseteq F(u_1, \dots, u_n) \supseteq F$$

Prop Let $K \supset F$, and $u, v \in K$. Then $F(u)(v) = F(u, v)$

$$K \supseteq F$$

$$K \supseteq F(u, v) \supseteq F$$

$$K \supseteq F$$

$$\underline{K \supseteq F(u) \supseteq F}$$

$$K \supseteq F(u)(v) \supseteq F(u) \supseteq F$$

Pf $F(u)(v) \supseteq F(u, v)$ because $F(u, v)$ is the smallest containing $F, u,$ and v .

$F(u, v) \supseteq F(u)(v)$ because $F(u, v)$ contains F and u , therefore $F(u, v) \supseteq F(u)$

$F(u, v)$ contains $F(u)$ and v , therefore $F(u, v) \supseteq F(u)(v)$

Th 11.10 Let $K = F(u_1, \dots, u_n)$

be a finitely generated extension.

Assume that all u_j are algebraic

Then K is finite-dimensional, therefore algebraic.
(Th 11.9)

The proof makes use of Th 11.4: In $F \subseteq K \subseteq L$, if $[L:K]$ and $[K:F]$ are finite, then so is $[L:F] = [L:K][K:F]$

Cor 11.11 If $L \supseteq K$ is algebraic and $K \supseteq F$ is algebraic, then $L \supseteq F$ is algebraic

Warning: an algebraic extension $\underbrace{F \subseteq K \subseteq L}$ may be not finite-dimensional.

However, finitely generated algebraic extension is finite-dimensional

Pf Let $u \in L$. Wanted: u is algebraic over F .

Since $L \supseteq K$ is algebraic, u is algebraic over K , meaning

$$a_0 + a_1 u + \dots + a_m u^m = 0, \quad a_i \in K$$

Since $K \supseteq F$, all a_i are algebraic over F .

$$L \supseteq \underbrace{F(a_0, \dots, a_m)}(u) \supseteq \underbrace{F(a_0, \dots, a_m)} \supseteq F$$

simple algebraic therefore finite-dimensional by Th 11.7 (3)

finite-dimensional by Th 11.10

finite-dimensional by Th 11.4 therefore algebraic by Th 11.9

Cor 11.12 Let $K \supset F$.

Let $E = \{u \in K \mid u \text{ is algebraic over } F\}$

If $u \in F$,
then u is a root
of $x - u \in F[x]$

Then ① E is a subfield of K

② E is an algebraic extension of F .

Pf ① implies ② immediately (by the definition of algebraic extension)

① If $u, v \in E$, then so are $u+v, uv, -u, -v$
are algebraic over F
 $u^{-1}, v^{-1} \in E$

Consider

$$K \supseteq F(u, v) \supseteq F$$

$F(u, v)$ is a finitely generated extension of F

Both u and v are algebraic, thus $F(u, v)$ is an algebraic extension by Th 11.10.

Since $u+v, uv, \dots \in F(u, v)$, all these elements are algebraic.

$$\begin{array}{l} \text{Ex } F = \mathbb{Q} \\ \sqrt{2} \quad x^2 - 2 \\ \sqrt[3]{2} \quad x^3 - 2 \\ f \in \mathbb{Q}[x] \\ f(\sqrt{2} + \sqrt[3]{2}) = 0 \end{array}$$

Example $F = \mathbb{Q}$, we have an extension $\mathbb{C} \supset \mathbb{Q}$

$\bar{\mathbb{Q}} = \{ u \in \mathbb{C} \mid u \text{ is algebraic over } \mathbb{Q} \}$ - the field of algebraic numbers,

$\bar{\mathbb{Q}} \supset \mathbb{Q}$ is an algebraic extension

not finite-dimensional

If $[\bar{\mathbb{Q}} : \mathbb{Q}] = n$, then every $u \in \bar{\mathbb{Q}}$ would be a root of a polynomial of degree at most n .

However, there exist irreducible polynomials, of arbitrary high degree (from Eisenstein criterion) from $\mathbb{Q}[x]$

A complex root $u \in \mathbb{C}$ of such a polynomial cannot be a root of a polynomial of smaller degree.